We start with the position vector of P:

$$r^P = (R + \ell\cos q)b_1 + \ell\sin q b_2 \qquad (2)$$

We next form the velocity vector for P:

$$v^P = -\ell\sin q(\omega + \dot{q})b_1 + [\ell\cos q(\omega + \dot{q}) + R\omega]b_2 \qquad (3)$$

where ω is the constant angular velocity of the unit vectors b_1 and b_2. Now, applying Kane's method, we find the coefficient of $\dot{q}$ in Eq. (3). This is

$$v_{\dot{q}} = \ell\sin q b_1 + \ell\cos q b_2 \qquad (4)$$

The analysis to this point has not introduced any linearization. It is appropriate at this point to do so. We linearize Eq. (4):

$$v_{\dot{q}} = -\ell q b_1 + \ell b_2 \qquad (5)$$

The velocity of P may now be linearized:

$$v^P = -\ell q\omega b_1 + [\ell(\omega + \dot{q}) + R\omega]b_2 \qquad (6)$$

Note that Eq. (5) cannot be obtained by finding the coefficient of $\dot{q}$ in Eq. (6). The nonlinear velocity is required to find the correct linearized partial velocity vector $v_{\dot{q}}$.

The next step is to use the linearized velocity to find the linearized acceleration vector for P:

$$a^P = -[2\ell\omega\dot{q} + (R+\ell)\omega^2]b_1 + \ell[\ddot{q} - \omega^2 q]b_2 \qquad (7)$$

The final step is to form the dot product $v_{\dot{q}} \cdot a^P$. This yields the equation of motion

$$\ell^2\ddot{q} = 0 \qquad (8)$$

Now, to apply Gibbs' method to this problem, the nonlinear acceleration vector must be formed, even to use the advantageous method of implicit differentiation. The coefficient of $\ddot{q}$ in Eq. (7) is ℓb_2, which differs from the correctly linearized partial velocity given in Eq. (5). If this vector is used to carry out the dot product which led to Eq. (8), we find instead

$$\ell^2\ddot{q} = \ell^2\omega^2 = 0 \qquad (9)$$

which differs from Eq. (8). Equation (8) would have resulted if we had formed the nonlinear acceleration, found the coefficient of $\ddot{q}$, and then linearized. The reader can certainly appreciate the additional labor required to carry out these steps. Thus, it appears that the connection between Kane's method and the Gibbs-Appell equation is not so simple as Desloge implied. Kane's method never required the formation of nonlinear acceleration vectors, and so represents a considerable savings in labor when one is solely interested in linearized equations.

References

[1]Desloge, E. A., "Relationship Between Kane's Equations and the Gibbs-Appell Equations," *Journal of Guidance, Control, and Dynamics*, Vol. 10, Jan-Feb. 1987, pp. 120-122.

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Comment on "Relationship Between Kane's Equation and the Gibbs-Appell Equations"

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IT is not surprising that one fundamental equation of dynamics can be derived from another. Thus, for example, Lagrange's equation and Hamilton's canonical equations can be derived from the simultaneous application of D'Alembert's principle and the principle of virtual work. Yet it does not seem inappropriate to identify Lagrange's equation and Hamilton's equations as separate entities. By Desloge's own derivation,[1] the Gibbs-Appell equation, as he calls Eq. (9) in his paper, *follows* from Kane's equation, Eq. (6) in the paper. Apparently it is Kane, and not Gibbs or Appell, who recognized the intermediate step of Eq. (6) as an entirely satisfactory termination point in deriving equations of motion (with no need to introduce the S function and proceed with the Gibbs-Appell equation).

In fact, one can show that the use of Kane's equation is operationally superior to the use of the Gibbs-Appell equation. This can be brought to light by performing the task of deriving equations linearized in certain variables. The importance and engineering significance of such linearization becomes apparent in formulating the equations of motion of flexible vehicles experiencing small elastic vibrations while undergoing large "rigid-body" motion. To clarify ideas, consider the simple example of Fig. 1.

AB and BC are two massless rigid rods of lengths R and L, respectively, hinged at B; AB is driven in a horizontal plane at the angular speed $\omega(t)$; the two rods are connected by a torsional spring of stiffness k, and P is a particle of mass m. It is desired to derive equations linear in θ. The steps involved in doing this using Kane's method and the Gibbs-Appell method are shown here, where all quantities linearized in θ are marked with a tilde:

Kane's method:

Nonlinear velocity:

$$v^P = \omega R a_2 + L(\omega + \dot{\theta})(-\sin\theta a_1 + \cos\theta a_2) \qquad (1)$$

Linearized partial velocity:

$$\tilde{v}^P_{\dot{\theta}} = -L\theta a_1 + L a_2 \qquad (2)$$

Linearized velocity:

$$\tilde{v}^P = \omega R a_2 + L(\omega + \dot{\theta})(-\theta a_1 + a_2) \qquad (3)$$

Linearized acceleration:

$$\tilde{a}^P = a_1[-\dot{\omega}L\theta - 2\omega L\dot{\theta} - \omega^2(R+L)] + a_2[\dot{\omega}(R+L) + L\ddot{\theta} - \omega^2 L\theta] \qquad (4)$$

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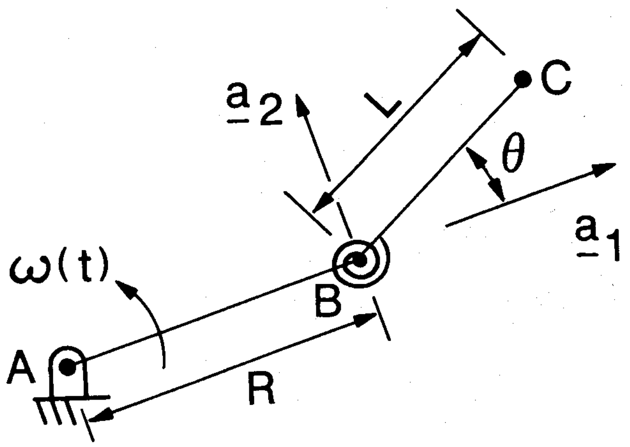

Fig. 1 Simple system undergoing small elastic motion in a rotating reference frame.

Kane's (linearized) equations of motion:

$$m\tilde{a}\cdot\tilde{v}^P_{\dot{\theta}}+k\theta=0 \tag{5}$$

where all nonlinear terms arising in the dot-product are ignored.

Gibbs-Appell method:

Nonlinear acceleration:

$$\begin{aligned} a^P = a_1[&-(\dot{\omega}+\ddot{\theta})L\sin\theta-(\omega+\dot{\theta})L\dot{\theta}\cos\theta \\ &-\omega^2R-\omega(\omega+\dot{\theta})L\cos\theta]+a_2[\dot{\omega}R+(\dot{\omega}+\ddot{\theta})L\cos\theta \\ &-(\omega+\dot{\theta})\dot{\theta}L\sin\theta-\omega(\omega+\dot{\theta})L\sin\theta] \end{aligned} \tag{6}$$

yields

$$\frac{\partial a^P}{\partial\ddot{\theta}}=-L\sin\theta a_1+L\cos\theta a_2 \tag{7}$$

Gibbs-Appell equation

$$\frac{\partial}{\partial\ddot{\theta}}[\tfrac{1}{2}ma^P\cdot a^P]+k\theta=0 \tag{8}$$

after implicit differentiation becomes

$$m\tilde{a}^P\cdot\frac{\partial\tilde{a}^P}{\partial\ddot{\theta}}+k\theta=0 \tag{9}$$

where all nonlinear terms arising in the dot-product are ignored. Now Eq. (9) is, of course, identical to Kane's equation, provided $\partial a^P/\partial\ddot{\theta}$ is linearized from Eq. (7), which requires that Eq. (6) be kept fully nonlinear in θ as is done here. This is precisely the point: to get correctly linearized equations via the Gibbs-Appell method, one must retain all nonlinearities up to the expression for acceleration—a cumbersome task not required in Kane's method, where one needs to retain nonlinearities only up to the velocity level. The additional labor involved with the Gibbs-Appell method over Kane's method can be so enormous with more complicated problems, such as in Ref. 2, as to make the analysis extremely unwieldly.

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[1]Desloge, E. A., "Relationship Between Kane's Equations and the Gibbs-Appell Equations," *Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 1, Jan.-Feb. 1987, pp. 120-122.

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Book Announcements

LEONDES, C.T., Editor, University of California at Los Angeles, ***Control and Dynamic Systems, Volume 24: Decentralized/Distributed Systems,*** Academic Press, New York, 1986, 362 pages, $57.50.

Purpose: This is the third in a series of three volumes on the subject of advances in techniques for the analysis and synthesis of decentralized or distributed control and dynamic systems.

Contents: A two-level parameter estimation algorithm for large-scale systems (M.P. Spathopoulis). Suboptimality bounds on decentralized control and estimation of large-scale discrete-time systems (M. Sinai). Decentralized control using observers (B. Shahian). System zeros in the decentralized control of large-scale systems (T.A. Kennedy). Direct model reference adaptive control for a class of MIMO systems (K.M. Sobel and H. Kaufman). Passive adaptation in control system design (D.D. Sworder and D.S. Chou). Index.

LEONDES, C.T., Editor, University of California at Los Angeles, ***Control and Dynamic Systems, Volume 25: System Identification and Adaptive Control,*** Academic Press, New York, 1987, 361 pages.

Purpose: This is the first in a series of three volumes devoted to the subjects of system parameter identification and adaptive control. The book is intended as a reference.

Contents: Uncertainty management techniques in adaptive control (H.V. Panossian). Multicriteria optimization in adaptive and stochastic control (N.T. Koussoulas). Instrumental variable methods for ARMA models (P. Stoica, B. Friedlander, T. Soderstrom). Continuous and discrete adaptive control (G.C. Goodwin, R. Middleton). Adaptive control: a simplified approach (I. Bar-Kana). Discrete averaging principles and robust adaptive identification (R.R. Bitmead, C.R. Johnson, Jr.). Techniques for adaptive state estimation through the utilization of robust smoothing (F.D. Groutage, R.G. Jacquot, R.L. Kirlin). Coordinate selection issues in the order reduction of linear systems (A.D. Doran).

ROBERTS, R.A., and **MULLIS, C.T.,** University of Colorado at Boulder, ***Digital Signal Processing,*** Addison-Wesley, New York, 1987, 578 pages.

Purpose: This text is used for a two-semester sequence on digital signal processing. The prerequisites for this material are a course in linear system theory including both discrete-time and continuous-time systems and some background in probability theory and random processes.

Contents: Digital signal processing. Discrete-time signals and systems. The z-transform. Fourier analysis of discrete-time signals and systems. Fast algorithms for the discrete Fourier transform. The approximation problem for digital filters. Least-squares filter design. Internal descriptions for digital filters. Finite length register effects in fixed point digital filters. Digital processing structures. Spectral estimation. Appendices. References. Suggested readings. Index.